AD-A095 757 NORTH CAROLINA STATE UNIV RALEIGH OPERATIONS RESEARCH F/G 12/2
OPTIMAL LINEAR APPROXIMATION IN PROJECT COMPRESSION.(U)
JAN 81 S E ELMAGHRABY, A M SALEM DAAG29-79-C-0211
UNCLASSIFIED NCSU-OR-171 ARO-16352.8-M NL

1 of 1
AD
A095757

END
DATE
FILMED
3-81
DTIC

LEVEL II 16352.8-M

OPERATIONS RESEARCH

NORTH CAROLINA STATE UNIVERSITY AT RALEIGH

① LEVEL II

OPTIMAL LINEAR APPROXIMATION
IN PROJECT COMPRESSION

S.E. Elmaghraby and A.M. Salem

OR REPORT NO. 171 JANUARY 1981

Classification Codes: Project Planning
Activity Networks
Chebychev Approximation

This research was partially supported by NSF Grant No. ENG-7813936 and ARO Contract No. DAAG29-79-C-0211.

OPTIMAL LINEAR APPROXIMATION IN PROJECT COMPRESSION

S.E. Elmaghraby and A.M. Salem
North Carolina State University
Raleigh, N.C. 27650

ABSTRACT

When the cost of reducing the duration of activities is convex and nonlinear, it may be advisable (to reduce the computing burden) to seek a "satisfising" answer, in which the project is "compressed" to a desired completion time with prespecified tolerable relative error. We treat the problem of constructing the optimal first degree interpolating linear spline that guarantees such maximal error, and consider various possible refinements.

OPTIMAL LINEAR APPROXIMATION IN PROJECT COMPRESSION

S. E. Elmaghraby and A. M. Salem
North Carolina State University
Raleigh, NC

INTRODUCTION

A significant recent trend in the study of activity networks (ANs) is the shifting of focus from insisting on achieving the optimum to requiring only approximations that deviate from the exact results by a known (perhaps predetermined) error; i.e., satisfising instead of optimizing. In the overwhelming majority of cases, asking for an approximation is more realistic than demanding the exact answer because of the very approximate nature of the input data.

It is a truism that project compression under linear or piecewise linear cost functions of the form $C(y) = b - ay$; $\ell \leq y \leq u$; $a,b > 0$ is considerably easier to resolve optimally than under nonlinear cost functions. The analysis of [2] and [3], carried under the simplifying assumption of quadratic cost functions, should amply demonstrate this fact, if such demonstration were needed! The natural question then is: What if $C(y)$ is not quadratic, though still convex decreasing as y increases from ℓ to u? For instance, suppose

$$C(y) = a/(b + ry) \quad ; \quad 0 \le \ell \le y \le u < \infty \quad ; \quad a,b,r > 0 \qquad (1)$$

What can be said about the optimum in this case?

One may wish to persist in applying the theoretical constructs of [2] and [3], which are indeed applicable _in toto_. Unfortunately, the Kilter Diagram (which is simply the plot of the derivative, -dC/dy versus the duration y) will now possess a nonlinear segment in region R, as shown in Fig. 1 for C(y) of (1), which would necessitate the solution of a system of nonlinear equations in the

Figure 1 approximately here

"flows" $\{f_{ij}\}$ at each iteration; an onerous task at best.

The other alternative is to approximate the cost function C(y) by a continuous piecewise linear and convex function (i.e., linear spline) that is "optimal" in some sense. There are two immediate questions that present themselves: The first is to define the criterion of optimality of the approximation, and the second is to define the sense of the approximation itself. This paper answers these two questions, then proceeds to demonstrate the application of these answers to achieve the desired approximation.

Although our discussion is couched throughout in the vernacular of ANs, the approach has broad applicability to a wide class of convex separable programming problems.

THE CRITERION

Let $C(y)$ be a nonlinear convex decreasing function on $[\ell, u]$. Let the points $y_0, y_1, \ldots, y_{n+1}$ satisfy $\ell = y_0 < y_1 < \ldots < y_n < y_{n+1} = u$ and let c_i, $1 \leq i \leq n + 1$ be arbitrary real numbers. Define

$$h(y) = c_{i-1} + (c_i - c_{i-1})(y - y_{i-1})/(y_i - y_{i-1}) \tag{2}$$

$$\text{for } y \varepsilon [y_{i-1}, y_i] \quad , \quad 1 \leq i \leq n + 1$$

Then h is a continuous piecewise linear function called a first degree spline with knots $\{y_i\}$, $0 \leq i \leq n + 1$ that joins the points (y_i, c_i), $0 \leq i \leq n + 1$. Since the knots y_0 and y_{n+1} are fixed, the problem is to choose the (variable) knots $y_1, y_2, \ldots, y_n$ and the ordinates $c_0, c_1, \ldots, c_{n+1}$ so that the corresponding spline h is as close to C as possible in the sense of the uniform norm; that is, we choose h to minimize the maximum error:

$$\text{min:} \quad \varepsilon = \max_{y \varepsilon [\ell, u]} |C(y) - h(y)|$$

$$= \| C - h \| \tag{3}$$

where h is a first degree spline with n-1 variable knots in the interval $[\ell, u]$. If such an h exists, it is called a best Chebychev (uniform, minimax) variable knot first degree spline (FDS) approximation to C on $[\ell, u]$. The value of ε is called the <u>error</u> or deviation of the approximation h.

Traditionally in spline approximation the values at the knots satisfy

$$c_i = C(y_i) \quad , \quad 0 \leq i \leq n + 1 \tag{4}$$

In this case, h is called a <u>first degree interpolating spline</u> (FDIS). If a spline h exists which minimizes (3) and satisfies (4), we call h a best Chebychev variable knot interpolating spline approximation to C on $[\ell, u]$. If we relax the restriction that $c_i = C(y_i)$, $0 \leq i \leq n + 1$, then we call h a best Chebychev <u>first degree approximating spline</u> (FDAS) with variable knots. Three important facts that are known about optimal Chebychev splines are worth recalling because of their relevance to our subsequent development:

1. The errors in all segments of the approximation are equal [5].
2. The maximum error in FDIS is exactly twice that in FDAS, which yields the least error.
3. For purposes of optimization, we may use any linear spline approximation that differs from the FDAS by a constant (the so-called "simple linear translation") [4].

Consequently we shall deal exclusively with FDIS. This choice will become even the more advisable as our discussion progresses.

In ref. [6] Salem gives elementary constructions, both graphical and analytical, for the determination of the optimal FDIS (and FDAS) that guarantee an error of value ε. (As a consequence, the number of knots is variable.) For our purposes, we briefly state the concepts underlying his graphical approach to the determination of the FDIS. The reader should have no difficulty in constructing the analytical procedure, which follows the graphical construction rather faithfully, and is stated in full detail in [6]. (In fact, we do just that for the specialized cost function in the example below.)

Suppose that $C(y)$ is plotted to some reasonable scale, as shown in Fig. 2. From the left-most point $(\ell; C(\ell))$, drop an ε and

Figure 2 approximately here

draw a tangent to the function $C(y)$; and from the point $(\ell; C(\ell))$, draw a parallel line to that tangent to intersect $C(y)$ at the point $(y^1, C(y^1))$. This is the first segment of the FDIS. The construction is now repeated starting with the point $(y^1, C(y^1))$; and is stopped when the upper limit u is reached. This "last segment" of the linear spline bears close scrutiny, since it may lead to further improvement in the approximation. We shall do this later, but for the moment it should be evident that the above construction yields a FDIS whose maximum deviation from $C(y)$ is at most ε. (The maximum deviation in the last segment may be strictly $< \varepsilon$.)

THE SENSE OF APPROXIMATION

Our concern is next focused on defining the sense of approximation that is appropriate for our application, and on determining the values ε_{ij};$(ij) \varepsilon A$, that realize it.

Recall that the problem of optimal project compression may be stated mathematically as (see, e.g., ref. [1]):

$$\underline{\text{Program P}}: \quad \text{Minimize } C(y) = \sum_{(ij)\varepsilon A} C_{ij}(y_{ij}) \tag{5}$$

$$\text{Subject to} \quad t_i - t_j + y_{ij} = 0 \quad ; \quad (ij)\varepsilon A \tag{6.1}$$

$$- t_1 + t_n = T_s \tag{6.2}$$

$$\ell_{ij} \leq y_{ij} \leq u_{ij} \tag{6.3}$$

Because of the difficulty in a frontal attack on Program P because of the nonlinearity of the objective function, we propose to approach it through approximating each individual $C_{ij}(y_{ij})$ by a FDIS, denoted by $h_{ij}(y_{ij})$. Thus we define the following surrogate program:

$$\underline{\text{Program S}}: \quad \text{Minimize } H(Y) = \sum_{(ij)\varepsilon A} h_{ij}(y_{ij}) \tag{7}$$

Subject to: (6.1) - (6.3)

Let v(P) and v(S) denote the minima of their respective programs. There are two possible criteria that measure the "goodness" of this approximation:

Criterion 1: Find ε_{ij}, $(ij)\varepsilon A$, such that the absolute deviation from the exact optimum is bounded by δ:

$$|v(S) - v(P)| \leq \delta \quad ; \quad \delta > 0 \tag{8}$$

Criterion 2: Find ε_{ij}, $(ij)\varepsilon A$, such that the relative deviation from the exact optimum is bounded by ω:

$$|v(S) - v(P)|/|v(P)| \leq \omega \quad ; \quad \omega > 0 \quad , \quad |v(P)| > 0 \tag{9}$$

In (8), the scalar δ is an "absolute" measure since it has the same dimension as the function C (say dollars), while the scalar ω of (9) is dimensionless since it represents the ratio of the (absolute) error of approximation to the minimal value of program P; typically $\omega \varepsilon [.01, .10]$. Clearly, Criterion 1 is useful when the analyst has a fairly precise idea about the value of v(P) on which he can base the value of δ. Criterion 2, on the other hand, is free of such prior knowledge, since it requires only the specification of a tolerable relative deviation from the optimum of program P (a "satisfising" relative error) whatever that value may be.

Consider Criterion 1: We wish to find ε_{ij} such that

$$|v(S) - v(P)| = v(S) - v(P) \leq \delta$$

The reader will now realize the added rationale for choosing the FDIS, which enables us to replace the absolute difference by the straight difference, since $h_{ij}(y_{ij}) \geq C_{ij}(y_{ij})$ for $\ell_{ij} \leq y_{ij} \leq u_{ij}$. $v(S) = \sum_{(ij)} h_{ij}(\hat{y}_{ij}) \geq \sum_{ij} C_{ij}(\hat{y}_{ij}) \geq \sum_{(ij)} C_{ij}(y^*_{ij}) = v(P)$, where $\hat{Y}$ and Y^* are the optimizing vectors for Programs S and P, respectively.

But

$$v(S) - v(P) = \min_{y\varepsilon\Omega} \sum_{(ij)} h_{ij}(y_{ij}) - \min_{y\varepsilon\Omega} \sum_{(ij)} C_{ij}(y_{ij})$$

where Ω is the space of feasible solutions defined by (6.1) - (6.3). If the optimum of program S is realized at $\hat{Y}$, and the optimum of program P is realized at Y^*, then for each activity (ij) the contribution to the difference is $h_{ij}(\hat{y}_{ij}) - C_{ij}(y^*_{ij})$, which is obviously $\leq \varepsilon_{ij}$. Consequently, we conclude that

$$|v(S) - v(P)| \leq |\sum_{(ij)} \varepsilon_{ij}| \leq \sum_{(ij)} |\varepsilon_{ij}|$$

Therefore, by taking $\varepsilon_{ij} = \delta/m$ inequality (8) will be satisfied, where m is the number of activities in the network, $m = |A|$.

Next, consider Criterion 2, which is the one adopted henceforth. By construction of the FDIS,

$$h_{ij}(y_{ij}) - C_{ij}(y_{ij}) \leq \varepsilon_{ij}$$

Summing overall $(ij)\varepsilon A$,

$$\sum h_{ij}(y_{ij}) \leq \sum C_{ij}(y_{ij}) + \sum \varepsilon_{ij}.$$

Minimizing both sides of the inequality results in

$$v(S) \leq v(P) + \sum \varepsilon_{ij} \tag{10}$$

which we rewrite as

$$v(S)/v(P) \leq 1 + \sum \varepsilon_{ij}/v(P) \quad ; \quad v(P) > 0 \tag{10'}$$

Actually, ε_{ij} should be written $(\varepsilon_{ij}|\omega)$, and v(S) should be written $(v(S)|\{\varepsilon_{ij}\})$, since ε_{ij} is indeed a function of ω, and v(S) is a function of the errors $\{\varepsilon_{ij}\}$. We forfeit such rigor of expression for the sake of clarity of exposition.

From the statement of Criterion 2, we have

$$v(S)/v(P) \leq 1 + \omega \tag{11}$$

Equating the r.s. of (10) and (11), and assuming for simplicity (see below) that $\varepsilon_{ij} = \varepsilon$, the same value for all activities, we conclude that

$$\varepsilon \leq \omega . v(P)/m \quad ; \quad m = |A|$$

Since the value of v(P) is not known, any lower bound on that value will yield a more "conservative" ε. Such a lower bound is immediately available in $\sum_{(ij)} C_{ij}(u_{ij})$, and we put

$$\varepsilon = \omega . C(U)/m \leq \omega . v(P)/m \tag{12}$$

where $C(U) = \sum_{(ij)} C_{ij}(u_{ij})$. This is the ε to be used in the determination of the Chebychev-optimal FDIS.

Once the approximate problem S is solved and the values $\hat{Y}$ and v(S) are in hand, one may obtain the a posteriori bounds

$$v(S) - \sum_{(ij)} \varepsilon_{ij} \leq v(P) \leq C(\hat{Y}) \tag{13}$$

The right inequality is evident from the fact that $\hat{Y}$ is a feasible point of P; the left inequality follows from (10).

Two Issues

Thus far we have left two questions unanswered; they are: (i) How does one cope with the last segment of the linear spline if it generates an error less than ε, and how does one use such a fact to refine the approximation? (ii) How would one proceed if one desires to weigh the various activities differently, as reflected in the permissible error ε_{ij}?

Rather than clutter this paper with a mass of symbolism and formulas that will undoubtedly mask the rather elementary nature of the response to these two issues, we prefer to respond within the context of a numerical example that is deliberately chosen for its simplicity and tractability. The reader should have no difficulty in extrapolating to more complex functions. Indeed, many of our formulas developed for this simple example are directly applicable to others, as we shall demonstrate.

NUMERICAL EXAMPLE

Perhaps the simplest nonlinear convex decreasing cost function is a quadratic with continuous derivative over $[0, \infty]$ of the form $\alpha + \beta(u - y)^2$. Consider a project network with such activity cost and other data given in Table 1 and Fig. 3.

Table 1 approximately here

Figure 3 approximately here

This problem was solved optimally by the methods of [ ]; the optimal compression function is shown in Fig. 4.

Figure 4 approximately here

Now it is desired, for purposes of illustration, to solve the problem by approximating the cost function $C_{ij}(y_{ij}) = \alpha_{ij} + \beta_{ij}(u_{ij} - y_{ij})^2$ using the algorithm for finding the best FDIS. The error ε is to be chosen to guarantee that the optimum value of the approximating program does not deviate from the optimum value of the original problem by more than 10% of the true optimum, i.e., we are adopting Criterion 2 with $\omega = 0.10$.

It is easy to see that the graphical construction of Fig. 2 can be translated into the following analytical steps:

1. Determine $\bar{y}^k$ (the point of maximum error in the kth segment of the approximation) from

$$[C(\bar{y}^k) + \varepsilon - C(y^{k-1})]/[\bar{y}^k - y^{k-1}] = C'(\bar{y}^k) \quad ; \qquad k = 1,2,\ldots \tag{14}$$

where $C'(\bar{y}^k) = dC/dy\big|_{y=\bar{y}^k}$. For the case $C(y) = \alpha + \beta(u - y)^2$, the above expression yields

$$\bar{y}^k = y^{k-1} + \sqrt{\varepsilon/\beta} \quad ; \quad y^0 = \ell \tag{14'}$$

and the slope of the line segment between y^{k-1} and y^k is given by

$$C'(\bar{y}^k) = -2\beta(u - y^{k-1} - \sqrt{\varepsilon/\beta}) \tag{15}$$

2. Determine y^k (the end point of the <u>kth</u> segment) from

$$[C(y^k) - C(y^{k-1})]/[y^k - y^{k-1}] = C'(\bar{y}^k) \tag{16}$$

For the above quadratic, this yields

$$y^k = y^{k-1} + 2\sqrt{\varepsilon/\beta} \tag{16'}$$

Finally, using (12) with $\omega = 0.10$, $m = 5$ and $C(U) = 11$ $(=\sum \alpha_{ij})$ we have $\varepsilon = 0.220$.

Substituting for this value of ε and the other network parameters, we immediately deduce the linear splines given in Table 2.

Table 2 approximately here

To help interpret this table, consider arc (2,3): The spline has four segments as shown in Fig. 5, with the given knots and slopes. The variable $y_{2,3}$ is now replaced by the four variables x_{12}, x_{13}, x_{14} and x_{15}, whose coefficients in the (linear) criterion are precisely the corresponding "slopes" of Fig. 5.

Figure 5 approximately here

Program P in the original variables $\{y_{ij}\}$ is now translated into the approximating linear Program S:

$$\begin{aligned}\text{Minimize } H(T_s) = 93 &- 7.062\ x_1 - 5.187\ x_2 - 3.310\ x_3 \\ &- 1.434\ x_4 - .25\ x_5 - .9062\ x_6 - 7.186\ x_7 \\ &- 5.310\ x_8 - 3.434\ x_9 - 1.560\ x_{10} - .323\ x_{11} \\ &- 5.062\ x_{12} - 3.186\ x_{13} - 1.310\ x_{14} \\ &- .188\ x_{15} - 7.062\ x_{16} - 5.187\ x_{17} \\ &- 3.310\ x_{18} - 1.434\ x_{19} - .25\ x_{20} \\ &- 7.062\ x_{21} - 5.187\ x_{22} - 3.310\ x_{23} \\ &- 1.434\ x_{24} - .25\ x_{25}\end{aligned}$$

Subject to:

$$\begin{aligned} t_1 - t_2 + x_1 + x_2 + x_3 + x_4 + x_5 &\le 0 \\ t_1 - t_3 + x_6 + x_7 + x_8 + x_9 + x_{10} + x_{11} &\le 0 \\ t_2 - t_3 + x_{12} + x_{13} + x_{14} + x_{15} &\le 0 \\ t_2 + x_{16} + x_{17} + x_{18} + x_{19} + x_{20} &\le T_s \ (= t_4) \\ t_3 + x_{21} + x_{22} + x_{23} + x_{24} + x_{25} &\le T_s \\ x_1 + x_2 + x_3 + x_4 + x_5 &\le 4 \\ x_6 + x_7 + x_8 + x_9 + x_{10} + x_{11} &\le 5 \end{aligned}$$

$$x_{12} + x_{13} + x_{14} + x_{15} \leq 3$$

$$x_{16} + x_{17} + x_{18} + x_{19} + x_{20} \leq 4$$

$$x_{21} + x_{22} + x_{23} + x_{24} + x_{25} \leq 4$$

$x_5 \leq .248$; $x_{11} \leq .31$; $x_{15} \leq .186$; $x_{20} \leq .248$; $x_{25} \leq .248$, and

$x_i \leq .938$, for all other i; $x_i \geq 0$, for all i.

This LP was solved for the three values of T_s given by the "breakpoints" in the optimal compression function of Fig. 4, namely T_s = 8, 6.75 and 4. The results are given in Table 3, which contains one additional result for comparison, namely the column labeled $L^*(T_s)$.

Table 3 approximately here

This column gives the optimal solution under the "crude" linear approximation of each function by the single line segment joining the two end points of $C_{ij}(y_{ij})$ (see, e.g., Fig. 5). The values of the absolute error between the "true" and "approximate" optima are given in column 5; and their relative magnitudes (to the "true" optima) are given in the last column. The latter are seen to be well within the 0.10 specified tolerable error. Their small values are a reflection of the conservatism induced by having based the value of ε on the lower bound C(U); see Eq. (12).

Table 3 contains one additional piece of information, namely the a posteriori bounds on the optimum $C^*(y)$; see Eq. (13). The bounds are rather good, and they do indeed contain the "true" value.

It is also instructive to compare the optimal activity durations under the three representations of $C_{ij}(y_{ij})$: The original quadratic function, the FDIS, and the "crude" linear approximation. This is given in Table 4.† It is evident that the differences among the activity durations remain significant throughout the ranges of T_s between the "Quadr." and "Linear Spline" on one hand and the "Crude Linear" on the other, with the difference between the first two being rather small.

Table 4 approximately here

We are now ready to respond to the two issues raised above.

The Last Segment and Improved Approximation

First, there is the question of the "last linear segment" of the spline that terminates at u. A glance at Fig. 5, for example, and some simple calculations reveal that this last segment (in the interval [2.814, 3]) has a maximum error of only .008649, a far cry

---

†The fourth column under each T_s, labeled "Improved Linear Spline", will be explained below.

from the allowable value of $\varepsilon = 0.220$. Consequently, we may obtain a smaller error ε throughout the range of y if we insist on adding one more restriction to the FDIS; namely, that it <u>terminates on the function $C_{ij}(y_{ij})$ at both ℓ_{ij} and u_{ij} while utilizing the same number of knots.</u> The existence of such spline is guaranteed by the finiteness of the range of y_{ij}'s.

Fortunately, for the quadratic cost function adopted in this example, the analytical determination of the optimum FDIS satisfying this additional condition is rather elementary. (This was another reason for the adoption of this particular cost function.)

Consider the first segment of the FDIS between y^0 and y^1. It is easy to see that (see Fig. 3):

$$\varepsilon^1 = \frac{C(y^1) - C(y^0)}{y^1 - y^0}(\bar{y}^1 - y^0) - [C(\bar{y}^1) - C(y^0)] \quad ;$$

$$y^0 = \ell \tag{17}$$

But, since $C(y) = \alpha + \beta(u - y)^2$, we have

$$C(y^1) - C(y^0) = \beta(y^0 - y^1)(2u - y^0 - y^1)$$

and similarly,

$$C(\bar{y}^1) - C(y^0) = \beta(y^0 - \bar{y}^1)\ (2u - y^0 - \bar{y}^1)$$

Substituting in the expression for ε^1, we obtain

$$\varepsilon^1 = \beta(y^0 - \bar{y}^1)(\bar{y}^1 - y^1)$$

But $\bar{y}^1$ is the point at which the tangent to C(y) is parallel to the line segment; i.e.,

$$\left.\frac{dC(y)}{dy}\right|_{y=\bar{y}^1} = -\,2\beta(u - \bar{y}^1) = \frac{C(y^1) - C(y^0)}{y^1 - y^0}$$

$$= -\,\beta(2u - y^0 - y^1) \tag{18}$$

yielding the relation between $\bar{y}^1$ and y^1 (recall that y^0 is fixed at ℓ):

$$\bar{y}^1 = (y^0 + y^1)/2$$

Consequently,

$$\varepsilon^1 = \beta\left(\frac{y^1 - y^0}{2}\right)^2$$

By a similar argument, we can easily deduce that, in general,

$$\varepsilon^k = \beta\left(\frac{y^k - y^{k-1}}{2}\right)^2 \quad ; \quad k = 2,3,\ .\ .\ .\ ,n+1$$

Now we impose the desired conditions, which are three: (i) $\varepsilon^k = \hat{\varepsilon}$, a constant for all k; (ii) $y^0 = \ell$; and (iii) $y^{n+1} = u$. They completely determine the value of ε. We proceed recursively:

$$\varepsilon^1 = \beta\left(\frac{y^1 - \ell}{2}\right)^2 = \hat{\varepsilon} \quad \Longrightarrow \quad y^1 = 2\sqrt{\hat{\varepsilon}/\beta} + \ell$$

$$\varepsilon^2 = \beta\left(\frac{y^2 - y^1}{2}\right)^2 = \hat{\varepsilon} \quad \Longrightarrow \quad y^2 = 2\sqrt{\hat{\varepsilon}/\beta} + y^1$$

$$= 2 \times 2\sqrt{\hat{\varepsilon}/\beta} + \ell$$

and in general $\varepsilon^k = \beta(\frac{y^k - h^{k-1}}{2})^2 \Longrightarrow y^k = k \times 2\sqrt{\hat{\varepsilon}/\beta} + \ell$

Finally, we have $y^{n+1} = u = (n+1) \times 2\sqrt{\hat{\varepsilon}/\beta} + \ell$

which yields

$$\hat{\varepsilon} = \beta[\frac{u - \ell}{2(n+1)}]^2 \tag{19}$$

This is the new value of ε to be used in the construction of the FDIS for this activity. Note that now the maximum error of approximation may be different for different activities.

To illustrate, consider once more activity (2,3) of Table 1 and Fig. 5: We have n = 3 knots, $\beta = 1$, u = 3 and $\ell = 0$; hence $\hat{\varepsilon} = (3/8)^2 = 9/64 \approx 0.140625$ (compare with the previous value of $\varepsilon = 0.220$). With this value of $\hat{\varepsilon}$, straightforward substitution in Eqs. (14), (15) and (16) yields the following Table 5:

Table 5 approximately here

The new knots and slopes for the various FDIS's of the project of Table 1 are given in Table 6. Of course, the number of x-variables for each activity remains the same as before, though their ranges and coefficients in the linear criterion are now different. The new

Table 6 approximately here

program, denoted by $\hat{S}$, was solved for the same three values of T_s as before, namely, $T_s = 8$, 6.75 and 4.00, and the optimum values are summarized in Table 7, together with the corresponding values for the "true" optimum and for the "original" FDIS. As to be expected, the absolute and relative errors are smaller than before. Moreover, the bounds on the optimum (also given in Table 7) are much sharper. (For comparison, see Table 3.) The optimum durations of the activities at these various target completion times have been included in Table 4 to facilitate comparison with other conditions.

Table 7 approximately here

The development thus far responds completely to the question raised concerning the "last segment of the FDIS" when the individual activity cost function is convex decreasing, quadratic and with continuous derivative for $y \varepsilon [\ell, \infty]$. Extension of the approach to quadratic cost functions that possess discontinuous derivative at $y = u$ (the case treated in ref. [2]) is straightforward and will not be elaborated upon here.

Such is not the case, however, for general cost functions that are convex decreasing in the interval $[\ell, u]$, such as the function $C(y)$ of (1). The reason is that now we may have to deal with nonlinear equations to determine the new value of $\hat{\varepsilon}$, though the logic of derivation implied by Eqs. (17) and (18) remains applicable in toto. In such instances, the application of simple iterative schemes yields the desired result in a few iterations. Indeed, bounds on the locations of the new knots $\{y^k\}_{k=1}^{n-1}$ are easily determined by applying the graphical (or analytical) construction of Eqs. (14)-(16) starting from the point $(u,C(u))$ rather than the point $(\ell,C(\ell))$.

The concepts of iteration are best illustrated by example. Consider again the simple network of Fig. 3, and suppose that the activity cost functions are given by Eq. (1). Elementary calculations (see also ref. [6], pp. 219-225) following the logic of Eqs. (14)-(16) yield:

$$\frac{C(\bar{y}^k) + \varepsilon - C(y^{k-1})}{\bar{y}^k - y^{k-1}} - C'(\bar{y}^k) = m^k$$

Substituting for $C(y) = a/(b + ry)$ in the above equation yields

$$\frac{a/(b + r\bar{y}^k) + \varepsilon - a/(b + ry^{k-1})}{\bar{y}^k - y^{k-1}} = - ar/(b + r\bar{y}^k)^2$$

or

$$[\varepsilon - a/(b + ry^{k-1})]\ (b + r\bar{y}^k)^2 + a/(b + r\bar{y}^k)$$

$$= -ar\ (\bar{y}^k - y^{k-1})$$

This is a quadratic in $(b + r\bar{y}^k)$ whose solution yields

$$\bar{y}^k = [- a - \{a^2 + a(b + ry^{k-1})(\varepsilon - a/(b + ry^{k-1})\}^{1/2}] \div$$

$$r(\varepsilon - a/(b + ry^{k-1})) - b/r$$

(choosing the root that yields $\bar{y}^k > y^{k-1}$).

Determine

$$m^k = C'(y)\Big|_{y=\bar{y}^k} = -\,ar/(b + r\bar{y}^k)^2$$

and compute y^k from

$$y^k = -\,(ar + m^k b\,(b + ry^{k-1}))/m^k r\,(b + ry^{k-1})$$

If $y^k \geq u$ stop and the last segment of the FDIS joins $C(y^{k-1}) \equiv C(y^n)$ and $C(u)$, i.e., put $y^{n+1} = u$.

To illustrate, assume the parameters are as given in Table 8, which is constructed so that the cost function satisfies the (reasonable) condition that would permit comparison later on with previous results, namely, that $C(y)$ passes by the points $C(\ell)$ and $C(u)$ of the quadratic cost function, $\alpha + \beta(u - y)^2$. It is easy to see that the cost function $C(y) = a/(b + ry)$ may be written, for simplicity, as $C(y) = a'/(1 + r'y)$ when both numerator and denominator are divided by b. Since $\ell_{ij} = 0$ and $\beta = 1$ for all activities in this example, this condition yields:

$$a' = C(0) \text{ and } r' = \frac{1}{u}\left[\frac{C(0)}{C(u)} - 1\right], \text{ resulting in}$$

$$a = \alpha C(0) \quad ; \quad b = \alpha \quad ; \quad r = u \tag{19}$$

Table 8 approximately here

For notational simplicity, we refer to the knots and errors that result from starting the spline at the point $(\ell, C(\ell))$ as resulting from <u>forward iteration</u>, and to the knots and errors that result from starting the spline at the point $(u, C(u))$ as resulting from <u>backward iteration</u>.

For illustration, consider once more activity (2,3): $C_Q(y) = 1 + (3 - y_{2,3})^2$, which yields (see (19) above) $C(y) = 10/(1 + 3y)$. Forward iteration is started with the original error of $\varepsilon_0 = 0.22$, which results in five segments as shown in Table 9. (Compare with only four segments of the quadratic cost function.)

Table 9 approximately here

Two remarks are relevant to Table 9. First, the unrestricted last segment of the FDIS "overshoots" the upper bound on the activity duration by a considerable amount. Therefore, it must be truncated, resulting in a much smaller error than ε_0.

Second, it is rather easy at this juncture to derive bounds on the locations of the knots through backward iteration [i.e., starting with spline at the point $(u, C(u))$ and proceeding "backwards" to the point $(\ell, C(\ell))$]. This is accomplished in Table 10 for the same value of $\varepsilon_0 = 0.22$.

Table 10 approximately here

The bounds on the location of the improved knots are shown in Fig. 6.

Figure 6 approximately here

The easiest iterative scheme is to put the new value of the error ε_1 equal to the average error in the previous iteration.† Therefore, put $\varepsilon_1 = \sum_k \varepsilon_0^k/5 = 0.192372$. The result is an improved spline (still of five segments) as shown in Table 11.

Table 11 approximately here

We iterate once more with $\varepsilon_2 = \sum_k \varepsilon_1^k/5 = 0.187169$, and obtain the values shown in Table 12.

---

†Alternative schemes, such as choosing the end-points of the just-determined intervals of the knots, were not systematically investigated.

Table 12 approximately here

The deviation from ε_2 in the last segment is 0.000755 or approximately 0.4%, and iteration may very well stop here. (In a computer program, one would pre-specify an "acceptable" deviation in the error at which iteration is terminated.) We decided to do one more iteration with $\varepsilon_3 = \sum_k \varepsilon_2^k/5 = 0.187018$. The results are shown in Table 13.

Table 13 approximately here

To be sure, the final knots are within their respective bounds established by the forward and backward iterations; see Fig. 6. (For example, $y^1 = .113978 \in [0.069616, 0.126214)$, $y^2 = 0.298232 \in [0.241950, 0.340470]$, etc.)

The complete (improved) FDIS's for all five activities are given in Table 14, obtained by similar (forward) iterations over the values of the error ε_{ij}. To be sure, the final errors achieved are all smaller than the starting error of $\varepsilon_0 = 0.22$, and they differ from activity to activity. The optimal solutions at the three "breakpoints" are given in Table 15, together with the respective bounds on the "true" optimum values based on them and on the value $\sum_{(ij)} \varepsilon_{ij} = 0.920068$.

Table 14 approximately here

Table 15 approximately here

The following remarks are pertinent.

It is evident that the cost of project compression is quite sensitive to the form of the assumed activity time-cost trade-off function, contrary to the common lore. For instance , at $T_s = 6.75$, the cost of the project varies from a low of 13.84 for the FDIS to the (steeply) convex cost function of (1); to 17.44 for the quadratic convex cost function $\alpha + \beta(u - y)^2$; to 17.70 for the FDIS to it; to 17.85 for its "improved" FDIS; to, finally, 27.25 for the "crude" linear approximation!

Though we have not conducted the requisite empirical investigation, it is our belief, on the basis of limited evidence, that the ratio of cost between the "crude" linear approximation and that of Eq. (1) will be significant (≥ 1.25). (Of course, this depends on the amount of compression required. In the above example, the duration was reduced by approximately 40%.)

The Differential Weighting of Activities

We remind the reader that this is the second question raised under "Issues" above. It is possible, in some application, that large errors will be tolerated in some activities but not in others. This differentiation among activities may be implemented by specifying a "tolerable" ratio between any pair of errors. This will introduce a slight modification to the development of Eqs. (10) through (12). In particular, we now write $\varepsilon_{ij} = a_{ij}\varepsilon$, $a_{ij} \geq 1$ and let $A = \sum_{(ij)} a_{ij}$. Then we have

$$v(S)/v(P) \leq 1 + A\varepsilon/v(p) \quad ; \quad v(P) > 0$$

which results in

$$\varepsilon \leq \omega . C(U)/A \quad \text{and} \quad \varepsilon_{ij} = a_{ij}\varepsilon \tag{20}$$

where, as before, $C(U) = \sum_{(ij)} C_{ij}(u_{ij})$.

As illustration, consider the simple network of Fig. 3 which has served as our vehicle thus far, and suppose that the weighting of the activities is in the ratio: $y_{1,2} : y_{1,3} : y_{2,3} : y_{2,4} : y_{3,4} = 1 : 2 : 1 : 1 : 3$. Then $A = 8$ and, for the parameters given in Table 1 and with $\omega = 0.10$, we obtain $\varepsilon = (0.1 \times 11) \div 8 = 0.1375$. Whence $\varepsilon_{1,2} = 0.1375 = \varepsilon_{2,3} = \varepsilon_{2,4}$; $\varepsilon_{1,3} = 0.2650$ and $\varepsilon_{3,4} = 0.4125$. From this point onwards, the procedure of spline approximation follows identical lines to those of either the FDIS or the improved FDIS, and will not be repeated.

REFERENCES

1. Elmaghraby, S. E. (1977) Activity Networks, John Wiley and Sons, New York.

2. Elmaghraby, S. E. and A. M. Salem (1980) "Optimal Project Compression Under Convex Functions, I", OR Report No. 157, North Carolina State University.

3. Elmaghraby, S. E. and A. M. Salem (1980) "Optimal Project Compression Under Convex Functions, II", OR Report No. 158, North Carolina State University.

4. Geoffrion, A. M. (1977) "Objective Function Approximations in Mathematical Programming", Math. Prog., 13, 23-37.

5. Gunanukorn, V. and D. F. McAllister (1979) "Application of Variable Knot First Degree Spline Approximation to Separable Programming", Department of Computer Science, North Carolina State University.

6. Salem, A. M. (1980) "Optimal Time Reduction in Activity Networks Under Convex Cost Functions", Unpublished Ph.D. Dissertation, North Carolina State University.

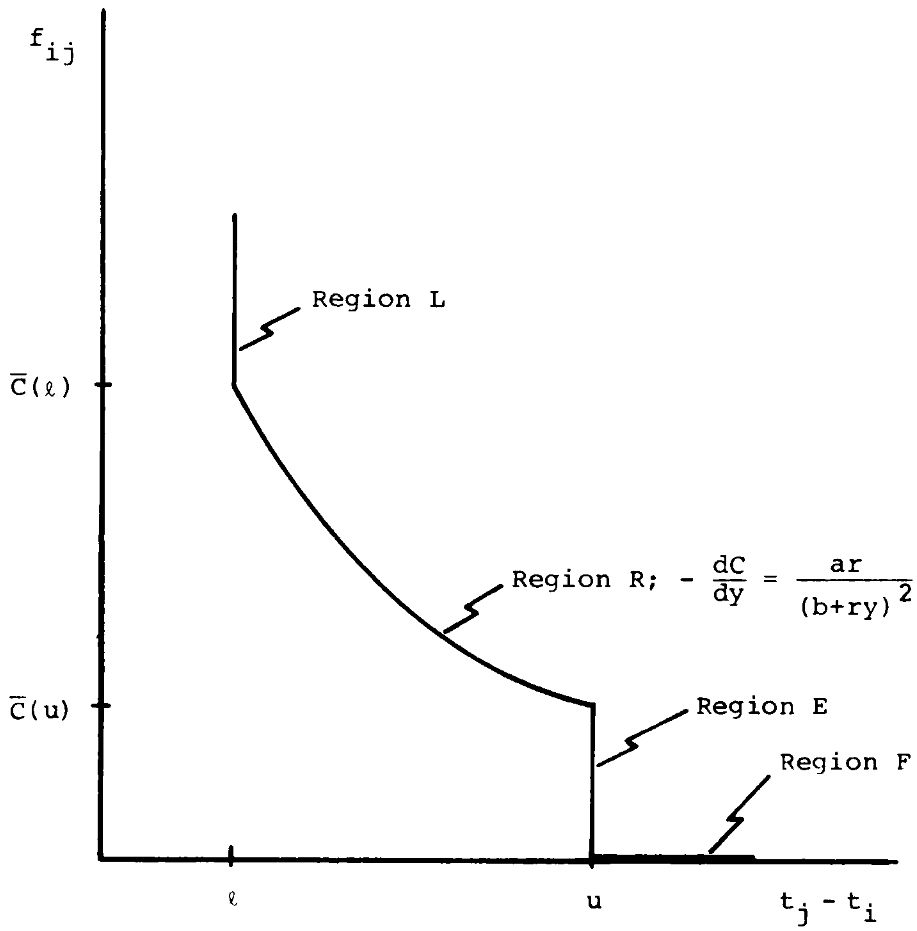

Figure 1 The KD for the Nonlinear Function $C(y)=a/(b+ry)$; $\ell \leq y \leq u$.

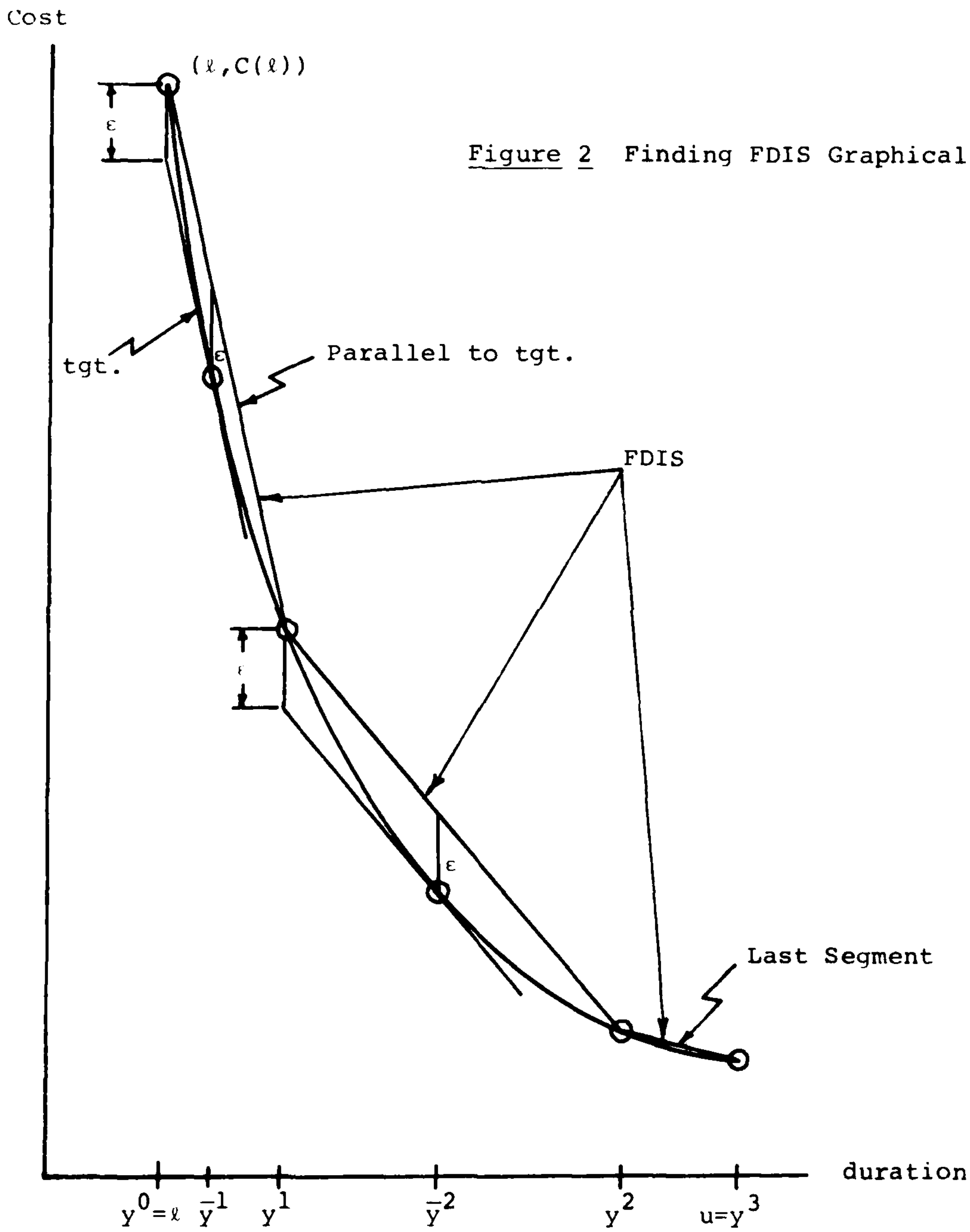

Figure 2 Finding FDIS Graphically

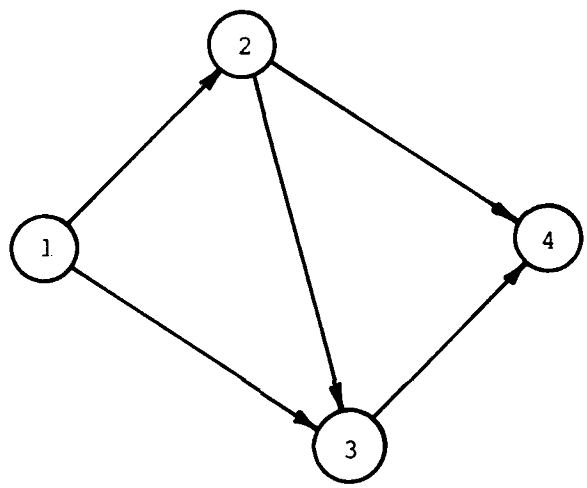

Figure 3

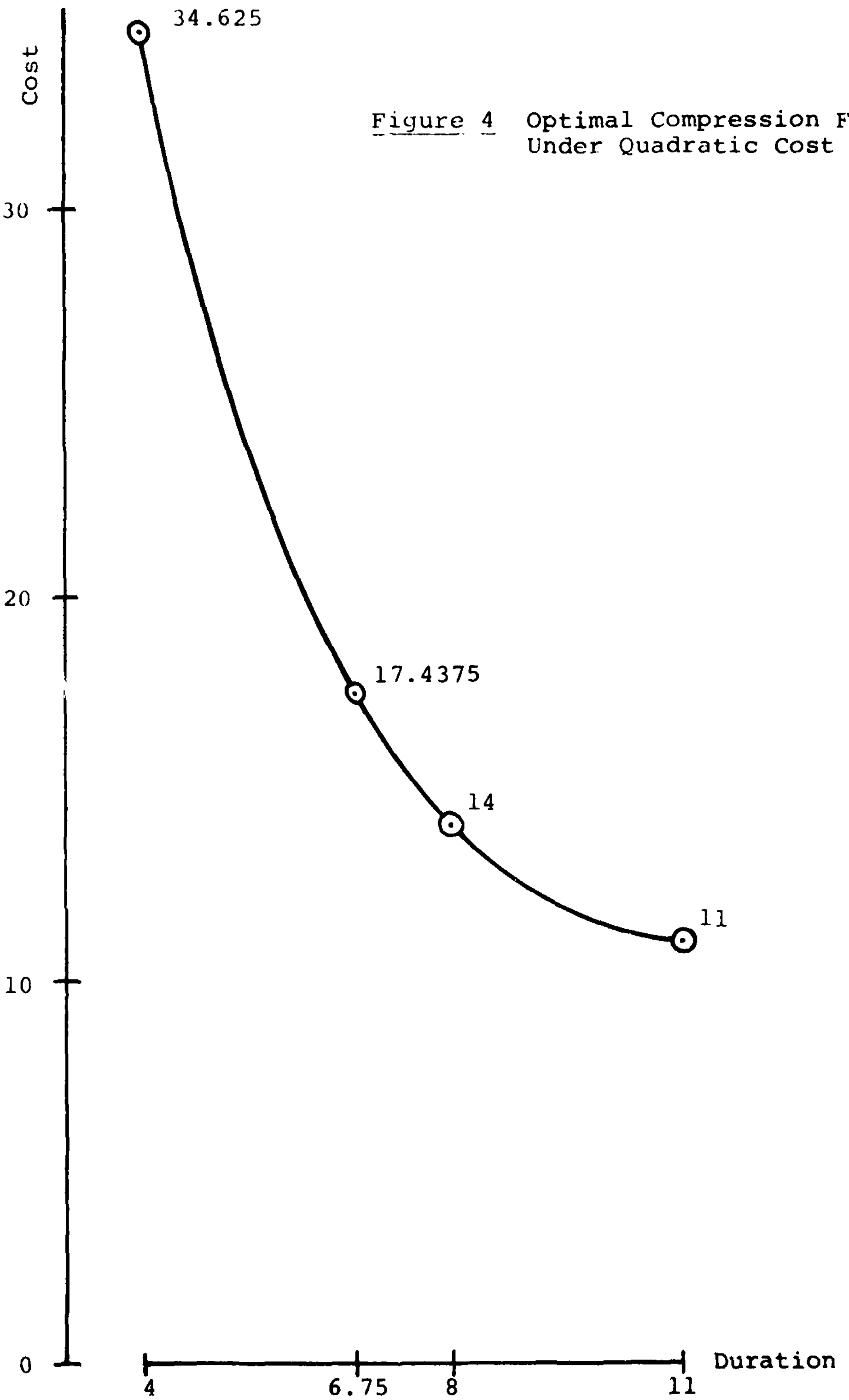

Figure 4 Optimal Compression Function
Under Quadratic Cost

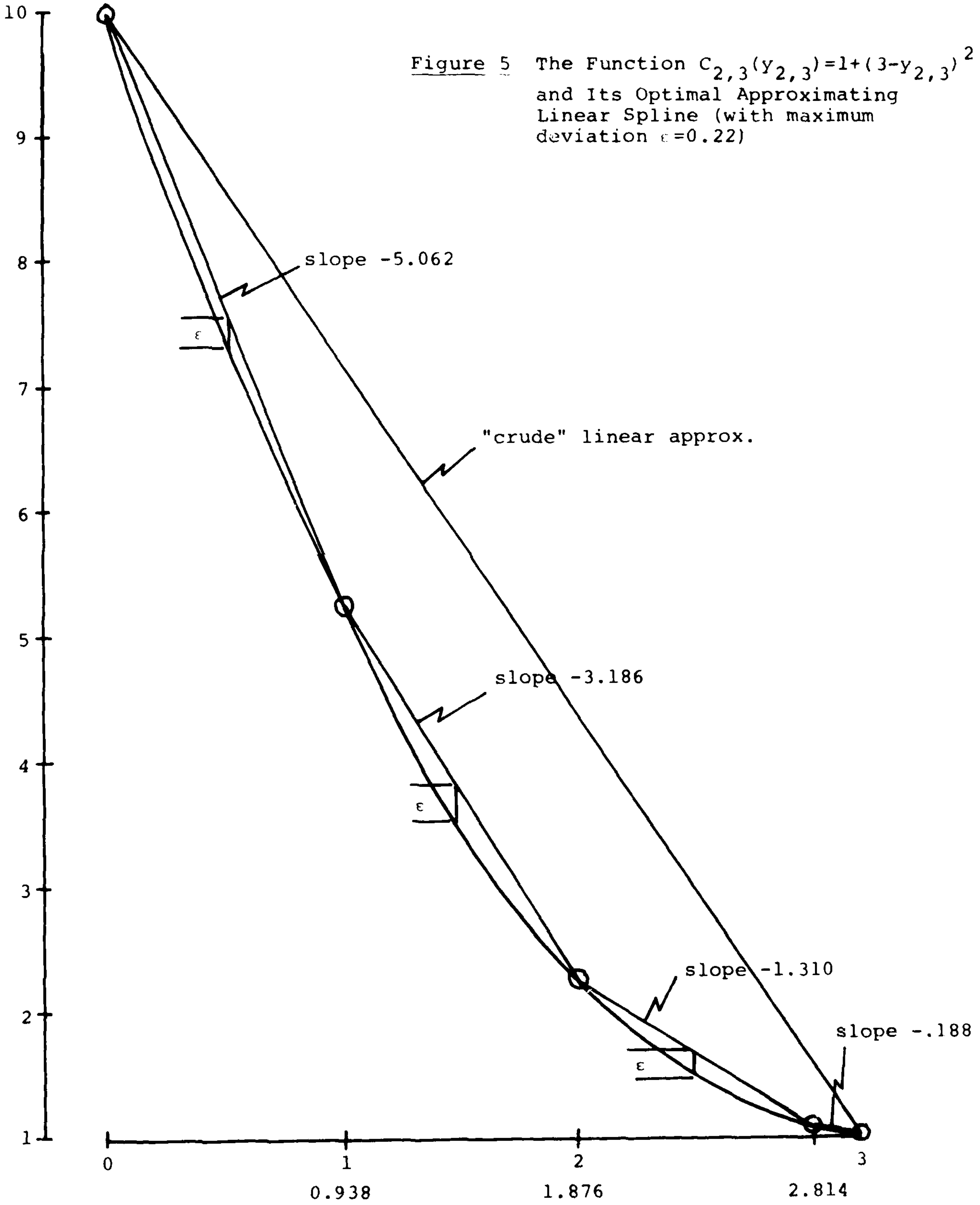

Figure 5 The Function $C_{2,3}(y_{2,3})=1+(3-y_{2,3})^2$ and Its Optimal Approximating Linear Spline (with maximum deviation $\varepsilon=0.22$)

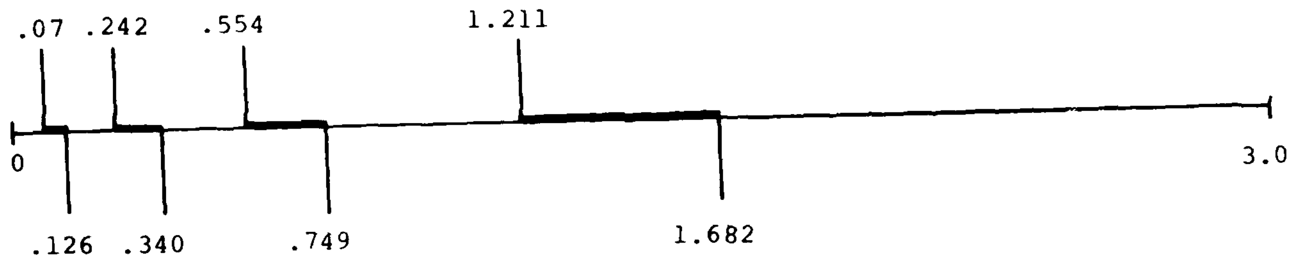

Figure 6 Bounds on $\{y^k\}$ for activity (2,3)
$C_{2,3}(y) = 10/(1+3y);\ 0 \le y \le 3$

Table 1

Activity	ℓ_{ij}	u_{ij}	$C_{ij}(y_{ij})$
(1,2)	0	4	$3 + (4 - y_{1,2})^2$
(1,3)	0	5	$4 + (5 - y_{1,3})^2$
(2,3)	0	3	$1 + (3 - y_{2,3})^2$
(2,4)	0	4	$2 + (4 - y_{2,4})^2$
(3,4)	0	4	$1 + (4 - y_{3,4})^2$

Table 2 Approximate FDIS's, $\varepsilon^k = 0.220$ (except in last segment)

Variable / Interval	y_{12}		y_{13}		y_{23}		y_{24}		y_{34}	
	x_i	slope	x_i	slope	x_i	slope	x_i	slope	x_i	slope
[0, .938]	x_1	-7.062	x_6	-9.062	x_{12}	-5.062	x_{16}	-7.062	x_{21}	-7.062
[.938, 1.876]	x_2	-5.187	x_7	-7.186	x_{13}	-3.186	x_{17}	-5.187	x_{22}	-5.187
[1.876, 2.814]	x_3	-3.310	x_8	-5.310	x_{14}	-1.310	x_{18}	-3.310	x_{23}	-3.310
[2.814, 3.752]	x_4	-1.434	x_9	-3.434	x_{15}	-0.188	x_{19}	-1.434	x_{24}	-1.434
[3.752, 4.690]	x_5	-0.250	x_{10}	-1.560	-	-	x_{20}	-0.250	x_{25}	-0.250
[4.690, 5.628]	-	-	x_{11}	-0.323	-	-	-	-	-	-
u_{ij}	4.00		5.00		3.00		4.00		4.00	
value at 0	19.00		29.00		10.00		18.00		17.00	
max. error in last segment	0.015376		0.024025		0.008649		0.015376		0.015376	

Table 3

T_s	$C^*(T_s)$ (1)	$H^*(T_s)$ (2)	$L^*(T_s)$ (3)	$\lvert C^*(T_s)-H^*(T_s)\rvert$	Posterior
8	14.00	14.3667	21.00	.3667	.0262
6.75	17.4375	17.6992	27.25	.2617	.0150
4	34.6250	35.0719	48.00	.4469	.0129

(1) Opt. solution of quadratic function

(2) Opt. solution of FDIS

(3) Opt. solution of "crude" linear approximation between ℓ_{ij} and u_{ij}.

	Interval Width
$13.2667 \le C^*(8) \le 14.0308$	0.7641
$16.5992 \le C^*(6.75) \le 17.5338$	0.9346
$33.9719 \le C^*(4) \le 34.7203$	0.7484

Table 4

	$T_s = 8$				$T_s = 6.75$				$T_s = 4$			
	Quadr.	Original Linear Spline	Crude Linear	Improved Linear Spline	Quadr.	Original Linear Spline	Crude Linear	Improved Linear Spline	Quadr.	Original Linear Spline	Crude Linear	Improved Linear Spline
(1,2)	3.0	3.124	4.0	3.2	$2\frac{2}{3}$	2.814	2.75	2.75	$1\frac{3}{8}$	1.186	1	1.6
(1,3)	5.0	5.0	5.0	5.0	$4\frac{1}{2}$	4.69	5	4.35	$3\frac{1}{8}$	3.062	4	3.2
(2,3)	2.0	1.876	1.0	1.8	$1\frac{3}{4}$	1.876	2.25	1.60	$1\frac{3}{4}$	1.876	3	1.6
(2,4)	4.0	4.0	4.0	4.0	4	3.936	4	4.00	$2\frac{5}{8}$	2.814	3	2.4
(3,4)	3.0	3.000	3.0	3.0	$2\frac{1}{4}$	2.060	1.75	2.40	$\frac{7}{8}$	0.938	0	0.8
Optimal Cost	14	14.3667	21	14.335	17.438	17.6992	27.25	17.8467	34.624	35.0719	48	34.9183

Table 5 Improved FDIS for Activity (2,3)

k	y^{k-1}	y^k	$\bar{y}^k$	$C'(\bar{y}^k)$
1	0	0.75	0.375	-5.25
2	0.75	1.50	1.125	-3.750
3	1.50	2.25	1.875	-2.250
4	2.25	3.00	2.625	-0.750
		$(= u_{2,3})$		

Table 6 Improved Spline Approximation

	$y_{1,2} = y_{2,4} = y_{3,4}$		$y_{2,3}$		$y_{1,3}$	
	Interval	Slope	Interval	Slope	Interval	Slope
	[0, 0.8]	-7.2	[0, 0.75]	-5.25	[0, 5/6]	-55/6
	[0.8, 1.6]	-5.6	[0.75, 15.0]	-3.75	[5/6, 10/6]	-7.5
	[1.6, 2.4]	-4.0	[1.50, 2.25]	-2.25	[10/6, 15/6]	-35/6
	[2.4, 3.2]	-2.4	[2.25, 3.0]	-0.75	[15/6, 20/6]	-25/6
	[3.2, 4.0]	-0.8	-	-	[20/6, 25/6]	-2.5
	-	-	-	-	[25/6, 5.0]	-5/6
Error ϵ_{ij}	0.160		0.140625		0.173611	

Table 7

T_s	Improved FDIS $H^*(T_s)$	"True" Opt. $C^*(T_s)$	Original FDIS $H^*(T_s)$	$\lvert C^*(T_s)=H^*(T_s)\rvert$	Posterior Improved w
8	14.3350	14.00	14.3667	0.3050	0.02178
6.75	17.8467	17.4375	17.6992	0.4092	0.02347
4	34.91833	34.6250	35.0718	0.2933	0.00847

Improved Bounds:	Interval Width
$13.5724 \le \hat{C}^*(8) \le 14.0800$	0.5076
$17.0525 \le \hat{C}^*(6.75) \le 17.5050$	0.4525
$34.1241 \le \hat{C}^*(4) \le 34.760$	0.6359

Table 8 Parameters of Network Under $C(y) = a/(b + ry)$

Activity	ℓ_{ij}	u_{ij}	a_{ij}	b_{ij}	r_{ij}
(1,2)	0	4	57	3	4
(1,3)	0	5	116	4	5
(2,3)	0	3	10	1	3
(2,4)	0	4	36	2	4
(3,4)	0	4	17	1	4

Table 9 Forward Spline for $\varepsilon_0 = 0.22$,
Activity (2,3)

k	y^k	$\bar{y}^k$	m^k	ε^k
0	0.00	-	-	-
1	0.126214	0.058052	-21.760562	ε_0
2	0.340470	0.223124	-10.765040	ε_0
3	0.748720	0.520535	- 4.571900	ε_0
4	1.681883	1.143343	- 1.528651	ε_0
5	3.00 (4.659654)	2.258459	- 0.496225	0.0816

y^k: kth knot value

$\bar{y}^k$: kth value of point of max. error

m^k: slope $C'(y)$ evaluated at $\bar{y}^k$

(The last number in parenthesis is the original knot if the value of y were not bounded.)

Table 10 Backward Spline with $\varepsilon_0 = 0.22$, Activity (2,3)

k	y^k	$\bar{y}^k$	m^k	ε^k
0	3.00	–	–	–
1	1.211247	1.935700	-0.647425	ε_0
2	0.554096	0.837438	-2.431836	ε_0
3	0.241950	0.381176	-6.529248	ε_0
4	0.069616	0.148133	-14.379588	ε_0
5	0.00 (-0.035459)	0.013118	-27.771252	0.355976

Table 11 Forward Spline for $\varepsilon_1 = 0.192373$, Activity (2,3)

k	y^k	$\bar{y}^k$	m^k	ε^k
0	0.00	-	-	-
1	0.115999	0.053678	-22.255222	ε_1
2	0.305046	0.202245	-11.620695	ε_1
3	0.644340	0.456683	- 5.340799	ε_1
4	1.348390	0.948921	- 2.027357	ε_1
5	3.00 (3.214741)	2.034312	- 0.594628	0.166357

Table 12 Forward Spline for $\varepsilon_2 = 0.187169$, Activity (2,3)

k	y^k	$\bar{y}^k$	m^k	ε^k
0	0	-	-	-
1	0.114035	0.05281	-22.352924	ε_2
2	0.298424	0.198285	-11.794044	ε_2
3	0.625641	0.445023	- 5.502014	ε_2
4	1.292741	0.915411	- 2.137625	ε_2
5	3.00 (3.005834)	1.994809	- 0.614978	0.186414

Table 13 Forward Spline for $\varepsilon_3 = 0.187018$, Activity (2,3)

k	y^k	$\bar{y}^k$	m^k	ε^k
0	0.00	-	-	-
1	0.113978	0.052806	-22.355783	ε_3
2	0.298232	0.198180	-11.799138	ε_3
3	0.625102	0.444686	- 5.506778	ε_3
4	1.291157	0.914452	- 2.140912	ε_3
5	3.00	1.993678	- 0.615576	ε_3
	(3.000008)			(Deviation $< 10^{-5}$)

Table 14 Improved FDIS for C(y) = a/(b+ry)

	$y_{1,2}$		$y_{1,3}$		$y_{2,3}$		$y_{2,4}$		$y_{3,4}$	
	Interval End Point	Slope	Interval End Point	Slope	Interval End Point	Slope	Interval End Point	Slope	Interval End Point	Slope
	0.17683	-20.49994	0.14226	-30.77704	.11398	-22.35578	.11080	-29.46939	.06435	-54.07978
	0.42446	-13.09111	0.32614	-21.86377	.29823	-11.79914	.26298	-19.31213	.15717	-33.20499
	0.78640	- 7.89719	0.56960	-15.04186	.62510	- 5.50678	.480	-12.03665	.29809	-19.04426
	1.34561	- 4.42587	0.90148	- 9.95556	1.29116	- 2.14091	.80473	- 7.03873	.52718	- 9.97727
	2.27659	- 2.24635	1.37041	- 6.28230	3.00	- 0.61558	1.32232	- 3.78528	.93687	- 4.60748
	4.00	- 0.99122	2.06371	- 3.73265			2.22222	- 1.81424	1.78216	- 1.76210
			3.15120	-2.05036			4.00	- 0.73469	4.00	- 0.49209
			5.00	-1.01235						
ε_{ij}	0.191676		0.179052		0.187018		0.163265		0.199057	

Table 15 FDIS Approximation for $C(y) = a/(b+ry)$

	$T_s = 8$	$T_s = 6.75$	$T_s = 4.00$
FDIS Cost	12.5729	13.8444	20.7589
$y_{1,2}$	4.0	2.750	1.8600
$y_{1,3}$	5.0	4.96784	3.1512
$y_{2,3}$	2.21784	2.21874	1.2912
$y_{2,4}$	4.0	4.00	2.1399
$y_{3,4}$	1.78216	1.78216	0.84876

$$11.6528 \leq C^*(8) < 12.3980$$

$$12.9243 \leq C^*(6.75) \leq 13.4917$$

$$19.8388 \leq C^*(4.0) \leq 21.6342$$

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 16352.8-M	2. GOVT ACCESSION NO. AD A095757	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Optimal Linear Approximation in Project Compression		5. TYPE OF REPORT & PERIOD COVERED Technical
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) S. E. Elmaghraby A. M. Salem		8. CONTRACT OR GRANT NUMBER(s) DAAG29-79-C-0211
9. PERFORMING ORGANIZATION NAME AND ADDRESS North Carolina State University Raleigh, NC 27650		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office Post Office Box 12211 Research Triangle Park, NC 27709		12. REPORT DATE Jan 81
		13. NUMBER OF PAGES 48
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

NA

18. SUPPLEMENTARY NOTES

The view, opinions, and/or findings contained in this report are those of the author(s) and should not be construed as an official Department of the Army position, policy, or decision, unless so designated by other documentation.

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

When the cost of reducing the duration of activities is convex and nonlinear, it may be advisable (to reduce the computing burden) to seek a "satisfising" answer, in which the project is "compressed" to a desired completion time with prespecified tolerable relative error. We treat the problem of constructing the optimal first degree interpolating linear spline that guarantees such maximal error, and consider various possible refinements.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

MED

8